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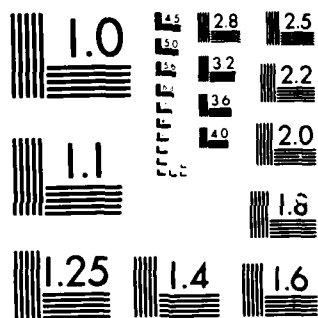
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# A FUZZY SET APPROACH TO VULNERABILITY ANALYSIS

Palmer R. Schlegel  
Ralph E. Shear  
Malcolm S. Taylor

December 1985

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gistic effects. Other contributions to uncertainty are of course present, starting from the target description through a myriad of subjective decisions and engineering judgements inherent in the vulnerability analysis.

For the purpose of completeness and comparison, a point estimate of the vulnerable area of the surrogate system will also be determined as it is currently done in vulnerability analysis. A point and interval estimate of the vulnerable area will be determined by a statistical procedure known as "bootstrapping." Finally, a fuzzy vulnerable area will be calculated and the different approaches compared.

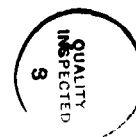
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## 1. INTRODUCTION

In defense preparedness studies, decisions requiring evaluation of the effectiveness of existing and conceptual weapon systems often arise. Such evaluations usually require some measure of the vulnerability of the subject systems relative to a specified scenario and mission profile. For example, if one is considering which weapon would be most effective against an enemy tank of the 1990's, a great deal of uncertainty surrounds the description of this future threat. Current vulnerability models require as input a detailed computer description of the system along with a configuration of its critical components.

Present methods of vulnerability analysis provide a point estimate; e.g., a probability of kill or a vulnerable area, with no corresponding measure of uncertainty that is inherent in the estimate. Attempts to model this uncertainty stochastically have been made by Dotterweich<sup>1</sup> and Taylor and Bodt.<sup>2</sup>

In this document, we will illustrate how vagueness or uncertainty can be modeled by fuzzy sets and how this uncertainty is reflected in the final measure of vulnerability. Our emphasis will be on the computation of vulnerable area of a surrogate system with uncertainty present in the cell probabilities. The uncertainty in the cell probabilities can be attributed to uncertainty in assigning component kill probabilities, to error in engineering judgement in determining critical areas of components and component kill criteria and to unknown synergistic effects. Other contributions to uncertainty are of course present, starting from the target description through a myriad of subjective decisions and engineering judgements inherent in the vulnerability analysis.

For purpose of completeness and comparison, a point estimate of the vulnerable area of the surrogate system will also be determined as it is currently done in vulnerability analysis. A point and interval estimate of the vulnerable area will be determined by a statistical procedure known as "bootstrapping." Finally, a fuzzy vulnerable area will be calculated and the different approaches compared.

## 2. DESCRIPTION OF A VULNERABLE AREA COMPUTATION

The models currently used for computing vulnerable area of armored vehicles require a number of detailed inputs, including a geometric description of the target specifying the location, thickness, and obliquity of armor plates and the location and size of major components. For a specified direction of attack, impact locations on the target are selected by superimposing a rectangular grid and choosing one point of impact within each grid cell. Then, at each point of impact, a ray is traced through the

<sup>1</sup> E. J. Dotterweich, "A Stochastic Approach to Vulnerability Analysis," ARBRL-TR-02578, Ballistic Research Laboratory, 1984

<sup>2</sup> M. S. Taylor and B. A. Bodt, "Construction of Approximate Confidence Intervals for Probability-of-Kill via the Bootstrap," ARBRL-TR-02570, Ballistic Research Laboratory, 1984

target model in the attack direction and a list of the components, barriers, empty spaces, and other structures encountered along the ray is compiled.

For each component encountered along the ray, a fragment mass and velocity together with an estimate of fragment shape is used to predict a hole area in the component. A kill criteria in the form of a hole of minimum cross-sectional area in a "sensitive region" of the component is required in order to regard the component as inoperable. If the predicted hole area exceeds the minimum hole area, then a probability of kill  $P_{k|h_j}$  is assigned to the  $j$ th component encountered by the  $i$ th ray. In determining probability of kill given a hit on a component, the vulnerability analyst estimates the "presented area" of the component and the presented area of the sensitive region of the component. The estimated probability of kill is taken as the ratio of the presented area of the sensitive region to the presented area of the component. (An overall component kill probability is obtained by summing the sensitive area from several aspects and dividing by the corresponding total component presented area.)

If several components are encountered along a ray, and determined to be killed, then the probabilities  $P_{k|h_j}$  are combined under an independence assumption to calculate a probability of kill,  $P_{k|h_i}$ , associated with the  $i$ th cell. This probability is then multiplied by the area of the cell,  $A_i$ , to yield a "cell vulnerable area"; cell vulnerable areas are then summed over all the cells in the grid to obtain an estimated target vulnerable area

$$A_v = \sum_i P_{k|h_i} A_i \quad (2.1)$$

for the given attack aspect. Hafer and Hafer <sup>3</sup> and Nail <sup>4</sup> provide further details concerning computation of vulnerable area.

### 3. ESTIMATION OF VULNERABLE AREA AND PROBABILITY OF KILL

Consider the conceptual item of military hardware illustrated in Figure 1. Without loss of generality, the component edges are assumed to be parallel with the superimposed 4x4 rectangular grid. The numbers in the  $i$ th cell represent the probability  $P_{k|h_i}$  that the target will be killed should a prescribed round of ammunition fired from a fixed range and orientation impact in that location.

The vulnerable area  $A_v$  for the target is calculated by

$$A_v = \sum_i P_{k|h_i} A_i = A \sum_i P_{k|h_i} \quad (3.1)$$

<sup>3</sup> T. F. Hafer and A. S. Hafer, "Vulnerability Analysis for Surface Targets (VAST)," ARBRL-TR-62154, Ballistic Research Laboratory, 1979.

<sup>4</sup> C. L. Nail, "Vulnerability Analysis for Surface Targets (VAST), Revision 1," Computer Sciences Corporation Report CSC-TR-52-5740, 1952.

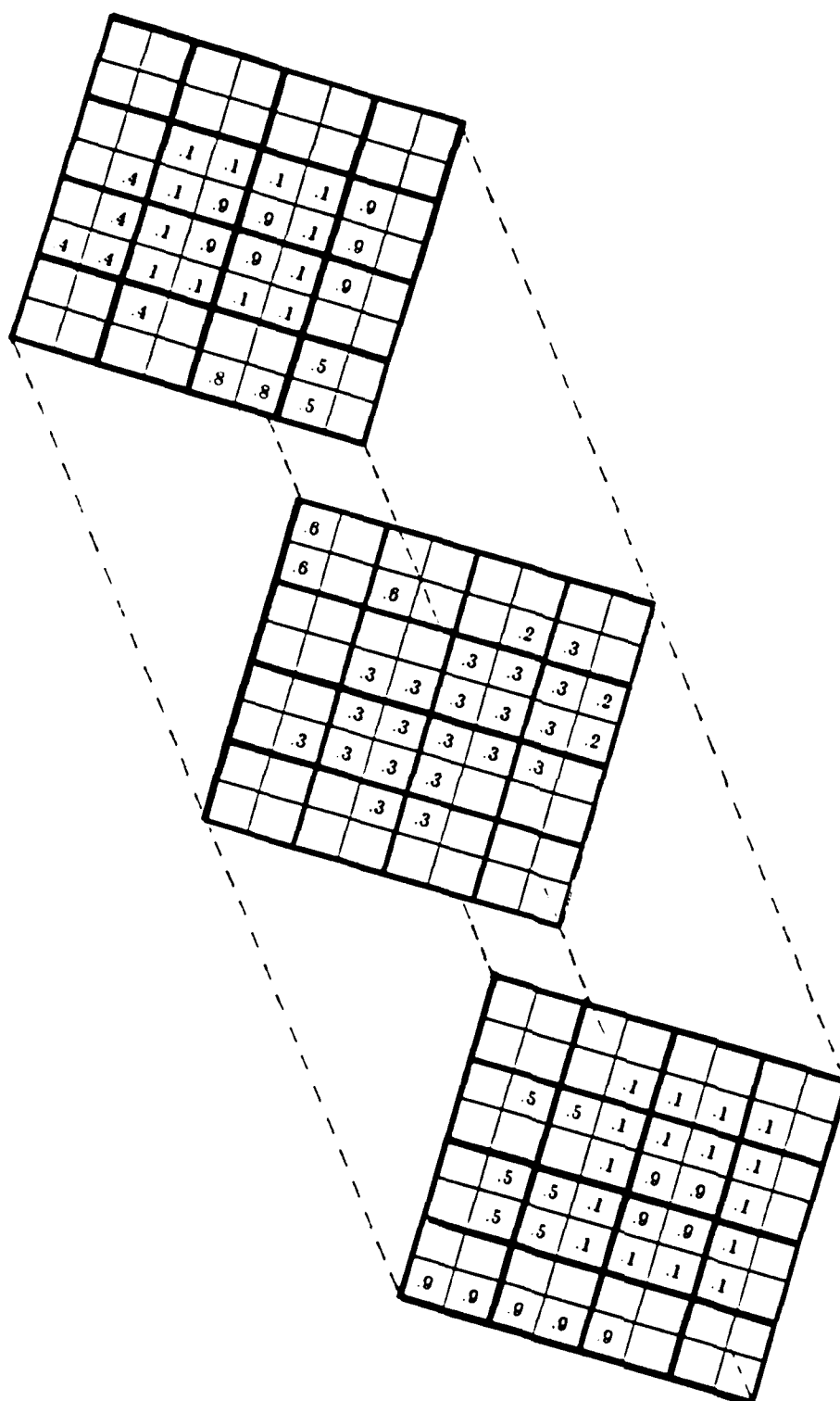


FIGURE 1. MULTI-LAYERED TARGET DESCRIPTION WITH ENCODED  $P_{k|h}$  VALUES.  
(Vacant cells correspond to  $P_{k|h} = 0$  )

where  $i$  is indexed over the cells of the grid. In this example, the cell areas  $A_i$  are identical and equal to  $A$ . The overall probability of kill  $P_k$  is

$$P_k = \frac{A_v}{A_p} \quad (3.2)$$

where  $A_p$  is the total presented area of the target. For the conceptual target shown in Figure 1, the exact values of vulnerable area and probability of kill are  $A_v = 105.1$  and  $P_k = 0.41$ .

In practice, the values  $P_{k|h_i}$  are unknown, and estimates  $\hat{P}_{k|h_i}$  are required to obtain the approximations

$$\hat{A}_v = A \sum_i \hat{P}_{k|h_i} \quad (3.3)$$

$$\hat{P}_k = \frac{\hat{A}_v}{A_p} \quad (3.4)$$

Determining valid  $P_{k|h_i}$  estimates is a persistent problem because of the difficulty in modeling the underlying damage mechanisms, and is an additional source of uncertainty. The bootstrap procedure to be introduced in Section 4 assumes the individual  $\hat{P}_{k|h_i}$ 's are accurately represented, and does not attempt to assign a component of error due to this uncertainty. This problem is however addressed in Section 7 where cell  $P_{k|h_i}$ 's are considered to be fuzzy numbers.

#### 4. THE BOOTSTRAP

The bootstrap is a technique for data analysis whose goal is to extract information from a set of data through repeated inspection. An expository article by Diaconis and Efron,<sup>5</sup> an intermediate-level paper by Efron and Gong,<sup>6</sup> or a more advanced paper by Efron<sup>7</sup> serve as an introduction to the bootstrap.

Suppose we want to estimate some attribute  $\theta$  of a population  $X = \{x_i\}_{i \in I}$ , where the index set  $I$  may be finite or infinite. To obtain an estimate  $\hat{\theta}$  of  $\theta$ , a random sample  $\{x_i\}_{i=1}^n$  is taken from  $X$  and a statistic  $\hat{\theta}(x_1, \dots, x_n)$  is evaluated. For example, if  $\theta = \mu$  (the population mean), then estimates

<sup>5</sup> P. Diaconis and B. Efron, *Computer-Intensive Methods in Statistics*, *Scientific American*, (1983), 116-130.

<sup>6</sup> B. Efron and G. Gong, *A Leisurely Look at the Bootstrap, the Jackknife, and Cross-Validation*, *The American Statistician*, 37(1983), 86-45.

<sup>7</sup> B. Efron, *Bootstrap Methods: Another Look at the Jackknife*, *Annals of Statistics*, 7(1979), 1-26.

$$\hat{\theta}(\cdot) = \frac{1}{n} \sum_{i=1}^n x_i \text{ or } \hat{\theta}(\cdot) = \text{median}(x_i) \quad (1.1)$$

are commonly used; if  $\theta = \sigma^2$  (the population variance), then

$$\hat{\theta}(\cdot) = \frac{1}{n(n-1)} \left\{ n \sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2 \right\} \quad (1.2)$$

is an appropriate statistic. If random samples  $\{x_i\}_{i=1}^n$  are repeatedly drawn from  $X$  and a statistic  $\hat{\theta}(\cdot)$  evaluated, a distinct value of  $\hat{\theta}(\cdot)$  may be determined for each sample drawn. These values are from the so-called sampling distribution of  $\hat{\theta}(\cdot)$ ;  $\hat{\theta}(\cdot)$  is itself a random variable under this sampling plan.

Knowledge of the sampling distribution of  $\hat{\theta}(\cdot)$  permits questions of accuracy to be answered, confidence intervals to be constructed and hypotheses to be tested. Under appropriate assumptions, and for some functional forms of  $\hat{\theta}(\cdot)$ , the sampling distribution may be determined analytically; however, in most instances it cannot. Furthermore, the option of repeatedly drawing random samples to construct an empirical approximation to the sampling distribution for  $\theta$  is usually not available. Typically, only a single sample of data  $\{x_i\}_{i=1}^n$  exists for analysis.

The essence of the bootstrap procedure for the application here is as follows: If samples with replacement from  $\{x_i\}_{i=1}^n$  are taken to generate "bootstrap" samples  $\{x_i^*\}_{i=1}^n$  and  $\hat{\theta}^* = \hat{\theta}(x_1^*, \dots, x_n^*)$  is determined a large number of times, the distribution of bootstrapped  $\hat{\theta}^*$  provides an approximation to the unknown sampling distribution of  $\hat{\theta}$ . (This procedure provides a Monte Carlo approximation to the distribution of the nonparametric maximum likelihood estimate of  $\theta$ .)

## 5. BOOTSTRAP DATA ANALYSIS

A random point of impact and its corresponding  $P_{k|h_i}$  were determined for each cell on the target in Figure 1. These sixteen values  $X = \{P_{k|h_i}\}_{i=1}^{16}$  form the data base for all subsequent analyses- conventional, bootstrap, and fuzzy. In this instance, the sixteen values formed the population that was sampled with replacement to determine bootstrap estimates  $A_v^*$  of the vulnerable area  $A_v$ . Estimate  $A_v^*$  was based on a sample equal to the number of cells in the overlaid grid. One thousand samples of sixteen were drawn from  $X$ , and one thousand estimates  $A_v^*$  computed. The approximation to the distribution of  $A_v$  is shown in Figure 2.

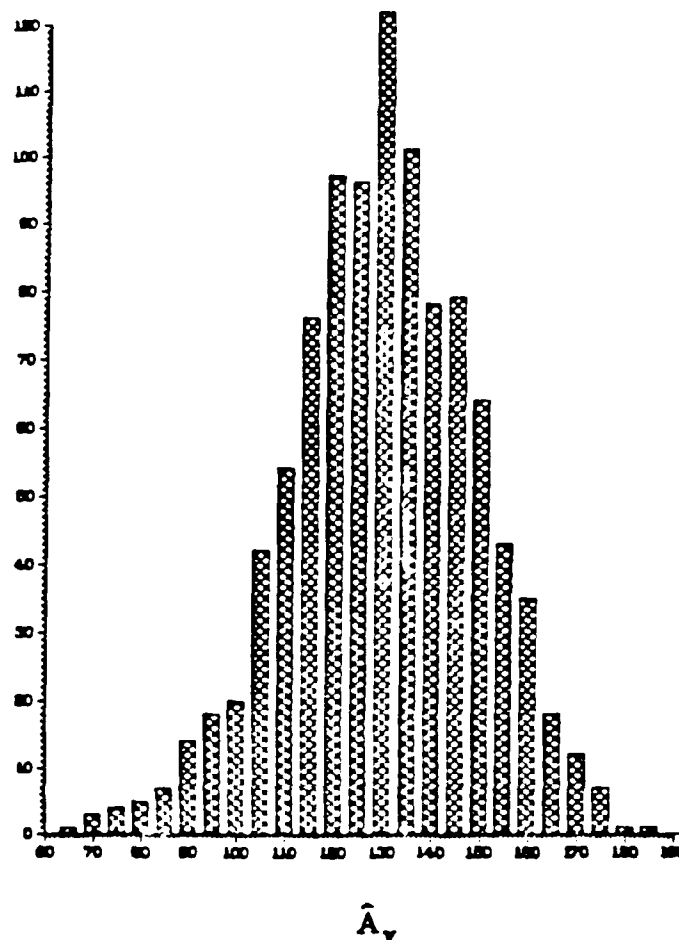


FIGURE 2. BOOTSTRAP ESTIMATE OF THE SAMPLING DISTRIBUTION OF  $\hat{A}_v$ .

The histogram in Figure 2 is a bootstrap estimate of the sampling distribution of  $\hat{A}_v$ . The median of this distribution, which we will take as an estimate of  $A_v$ , is  $A_v^* = 132.8$  (compared to the true vulnerable area  $A_v = 105.1$ ). An approximate central 90% confidence interval for  $A_v$  based on the bootstrap is [99.3, 163.6].

In this example, the median of the bootstrap distribution is considerably larger than the true vulnerable area. This is because the sample  $X$  which forms the basis for all the bootstrap calculations estimates  $A_v$  to be 133.2; i.e., the procedure operates on a sample providing a biased estimate of  $A_v$ . The bootstrap has no way to compensate for an unknown initial bias, and in this instance led to a bootstrap estimate  $A_v^* = 132.8$ . This bias has no effect on the estimate of variance of  $\hat{A}_v$  since the error is in location. The location error can be controlled in this application by using a finer grid on the target.

## 6. VAGUENESS IN VULNERABILITY ANALYSIS

As indicated in Section 2, to perform a vulnerability assessment of a given system, it is often necessary to develop an elaborate computer description. Information about the system may be varied in form and in abundance, ranging from intelligence data to fully documented commercial products. The system may be foreign or, in some instances, conceptual. One of the most subjective parts of the assessment involves what the vulnerability analyst may refer to as "engineering judgement." Knowledge of a component's function, for example, may cause the analyst to predict its size and location in a system for which little information is available. The analyst may be called upon to judge armor thickness, material and thickness of component structures, and to specify kill criteria for individual components, as well as their sensitive regions. The analyst assesses the effect of different types and degrees of damage to a system and its components and infers the systems "loss of function." Given damage to a component or subsystem, the analyst may infer a resulting loss of mobility, firepower, etc. The meaning of some of these terms and appropriate measures for their quantification are vague or imprecise; even vulnerability analysts differ as to the precise meanings and interpretations.

At many places in a vulnerability analysis, the analyst is required to act upon information which is vague or imprecise and, in some cases, where information is not available at all. Some of the uncertainty present in the process is not due to randomness and therefore probability theory may not provide a completely satisfactory modeling tool. The theory of fuzzy sets provides a rationale for dealing with sources of uncertainty which are non-random. A brief introduction to fuzzy sets is given in Appendix A. In the following section we illustrate the application of fuzzy sets to vulnerable area calculation.

## 7. CELL PROBABILITIES AS FUZZY NUMBERS

The vulnerable area model VAST requires as part of its input data a collection of rays corresponding to fragment trajectories for given angles of attack. These rays are represented by lists of components, armor and barriers that would be encountered by the fragments. Along each ray or trajectory the mass and velocity of the fragment as it exits obstacles it encounters is computed. For an initial fragment mass, velocity and shape at the target surface, the residual mass and velocity are calculated from the so-called THOR<sup>8</sup> equations. As the fragment encounters a particular component, a hole area is estimated in that component.

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<sup>8</sup> Project THOR, "The Resistance of Various Metallic Materials to Perforation by Steel Fragments, Empirical Relationships for Fragment Residual Velocity and Residual Weight," Project THOR Tech Report No. 47, Ballistic Analysis Laboratory, Inst. for Cooperative Research, The Johns Hopkins University, April (1961).

As indicated previously, there is a great deal of vagueness and subjectivity involved in the target description, material specification and thickness, component  $P_{k|h_0}$ , evaluation and kill criteria. Methods currently in use for vulnerability assessment, including the bootstrap procedure discussed in Section 4, do not take these uncertainties into account. Assumptions regarding the fragment shape, barrier material and thickness dictate that the hole estimate is actually a fuzzy number.

Incorporating fuzziness into both the kill criteria and the component conditional kill probability results in the following fuzzy implication: "If the cross-sectional hole area in the component is about  $\xi \text{ cm}^2$  or greater, then the component conditional kill probability  $P_{k|h_0}$  is B," where B is now a fuzzy number. Thus, we have a fuzzy hole area estimate and a fuzzy implication relating kill criteria with kill probability. If these were all "crisp," as assumed in the current method for vulnerability assessment, one need only compare the predicted hole area with that of the kill criteria. If the predicted hole area is greater than the required value, then a probability of kill is assigned. This is described by the syllogism

$$\begin{array}{l} \text{If } h \text{ is } A, \text{ then } p \text{ is } B \\ h \text{ is } A \\ \hline p \text{ is } B \end{array}$$

However, when A and B are fuzzy and we have  $h \text{ is } A^*$ , then we must use the *compositional rule of inference* or the *generalized modus ponens of fuzzy logic*<sup>9</sup> to obtain the conclusion. This can be represented by the syllogism

$$\begin{array}{l} \text{If } h \text{ is } A, \text{ then } p \text{ is } B \\ h \text{ is } A^* \\ \hline p \text{ is } B^* \end{array}$$

The resulting fuzzy component conditional kill probabilities, when combined, result in a fuzzy cell probability and finally a fuzzy vulnerable area estimate, reflecting the uncertainty in the process.

Zadeh<sup>10</sup> defined "If A then B" as a special case of the fuzzy implication "If A then B else C." Notationally, we represent If A then B else C as  $(A \times B) \cup (-A \times C)$  where "-" indicates negation and  $\times$  is the cartesian product of the two fuzzy sets. If C is the empty set, then If A then B is written  $(A \times B)$ . Recalling that the cartesian product of two fuzzy sets A and B is given by

$$\mu_{A \times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}. \quad (7.1)$$

<sup>9</sup> L. A. Zadeh, "Outline of a New Approach to the Analysis of Complex Systems and Decision Processes," *IEEE Trans. on Systems, Man and Cybernetics*, SMC-8(1), (1978).

<sup>10</sup> *ibid.*

we can derive a relation between the hole area and the kill probability, namely

$$\mu_R(h, p) = \min\{\mu_H(h), \mu_{P_{k|h_u}}(p)\} \quad (7.2)$$

where  $R = H \times P_{k|h_u}$ .

*Example:* If  $H = 0.1/h_1 + 0.5/h_2 + 0.7/h_3 + 1/h_4$  where  $h_i < h_j$  if  $i < j$ , and  $P_{k|h_u} = 0.1/p_1 + 0.5/p_2 + 1/p_3 + 0.5/p_4 + 0.1/p_5$ , then  $\mu_R(h, p)$  is given by

h \ p	$\mu_R(h, p)$				
	p <sub>1</sub>	p <sub>2</sub>	p <sub>3</sub>	p <sub>4</sub>	p <sub>5</sub>
h <sub>1</sub>	0.1	0.1	0.1	0.1	0.1
h <sub>2</sub>	0.1	0.5	0.5	0.5	0.1
h <sub>3</sub>	0.1	0.5	0.7	0.5	0.1
h <sub>4</sub>	0.1	0.5	1.0	0.5	0.1

Furthermore, if we assume that the hole estimate  $H_1$  is given by

$$H_1 = 0.3/h_1 + 1/h_2 + 0.5/h_3 + 0.1/h_4,$$

then we would like to combine this estimate with that given by the fuzzy implication to infer something about  $P_{k|h_u}$ . To proceed further, we need to invoke the compositional rule of inference.

Given universes  $U$  and  $V$ , a fuzzy set  $A$  on  $U$  and a fuzzy relation  $R$  on  $U \times V$ , characterized by  $\mu_A(u)$  and  $\mu_R(u, v)$  respectively, the compositional rule of inference states that the result will be a fuzzy set  $B$  on  $V$  defined by

$$\mu_B(v) = \bigcup_u [\mu_A(u) \cap \mu_R(u, v)] \quad (7.3)$$

and written as  $B = A \circ R$ .

Continuing the above example, we have

$$H_1 \circ R = B$$

with B denoting the fuzzy set resulting from this composition. Formally,

$$\begin{aligned}\mu_B(p_1) &= \cup \{ \mu_{H_1}(h_i) \cap \mu_R(h_i, p_1) \} \\ &= \cup \{ 0.3 \cap 0.1, 1.0 \cap 0.1, 0.5 \cap 0.1, 0.1 \cap 0.1 \} \\ &= \cup \{ 0.1, 0.1, 0.1, 0.1 \} \\ &= 0.1.\end{aligned}\tag{7.4}$$

Similarly,  $\mu_B(p_2) = \mu_B(p_3) = \mu_B(p_4) = 0.5$ , and  $\mu_B(p_5) = 0.1$ ; hence  $H_1 \circ R = B$ .

If  $H_1 = H$ , and both are non-fuzzy, then

$$H \circ R = P_{k|h_j}\tag{7.5}$$

in agreement with the classical modus ponens. With  $H_1 \neq H$  and fuzzy, we have a fuzzy version of the classical modus ponens with the important difference that  $H_1$  and  $H$  can be different. Thus, in the above example, we have a hole estimate which is different, in general, from the required kill criterion.

Similar results can be obtained for each component encountered along the fragment trajectory. These kill probabilities must be ultimately aggregated into a cell kill probability. As mentioned in Section 2, current methods include combining component kills as independent events; however, the problem of aggregation of these probabilities remains open. Some methods of aggregating fuzzy sets such as union and intersection are presented in Appendix A. Zimmerman and Zysno<sup>11</sup> have investigated several other methods. For vulnerable area calculations, aggregation of the fuzzy sets requires further research; therefore, in the following we shall assume that some form of aggregation has been done, and the resulting probability of kill associated with each cell,  $P_{k|h_j}$ , is a fuzzy number.

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<sup>11</sup>H.J. Zimmerman and P. Zysno, "Latent Connectives in Human Decision Making," *Fuzzy Sets and Systems* 4(1980), 87-51.

## 8. FUZZY ARITHMETIC

In this section, we consider  $P_{k|h_1}$  to be a fuzzy number and  $A$  to be a crisp number; to simplify the notation we will denote  $P_i = P_{k|h_1}$ . Equation (3.1) will then be evaluated by fuzzy arithmetic to obtain a fuzzy number

$$A_v = A \sum_i P_i. \quad (8.1)$$

Let  $\mu_{P_1}(x)$  and  $\mu_{P_2}(y)$  be normalized membership functions of  $P_1$  and  $P_2$  with support in  $[0,1]$ . The definition of the sum of two fuzzy numbers  $P_1 + P_2$  is given by

$$\mu_{(P_1 + P_2)}(t) = \max_{x+y=t} \left[ \min(\mu_{P_1}(x), \mu_{P_2}(y)) \right]. \quad (8.2)$$

We will choose our class of membership functions to be continuous and symmetric about a point  $x_0$  such that

$$\mu_P(x) = \begin{cases} \text{monotone nondecreasing} & x \leq x_0 \\ \text{monotone nonincreasing} & x \geq x_0 \end{cases}. \quad (8.3)$$

In particular, consider the following two functions:

$$\mu_{P_1}(x) = \begin{cases} 0 & x \leq x_1 \\ f(x) & x_1 < x \leq x_2 \\ 1 & x_2 < x \leq x_3 \end{cases} \quad (8.4)$$

and

$$\mu_{P_2}(y) = \begin{cases} 0 & y \leq y_1 \\ g(y) & y_1 < y \leq y_2 \\ 1 & y_2 < y \leq y_3 \end{cases} \quad (8.5)$$

where the functions  $\mu_{P_1}(x)$  and  $\mu_{P_2}(y)$  are symmetric about  $x_3$  and  $y_3$ , respectively. Dubois and Prade<sup>12</sup> show how to operate on fuzzy numbers of this form in an efficient manner. In Appendix B a geometric interpretation useful in evaluating (8.2) for our specific choice of membership function is detailed.

The product of two fuzzy numbers  $A$  and  $P$  is defined as

$$\mu_{A \cdot P}(t) = \max_{xy=t} \left[ \min(\mu_A(x), \mu_P(y)) \right]. \quad (8.6)$$

<sup>12</sup>D. Dubois and H. Prade, "Operations on Fuzzy Numbers," *Int. J. Systems Science*, 9(6), (1978) 618-626

In this instance,  $A$  is considered a crisp number, which has as its membership function

$$\mu_A(x) = \begin{cases} 1 & x = A \\ 0 & \text{otherwise.} \end{cases} \quad (8.7)$$

Since, for a given  $t$ ,  $\mu_A(x) = 0$  for  $x \neq A$ , the only value under consideration is  $x = A$ . This implies

$$\begin{aligned} \mu_{A \cdot P}(t) &= \max_{Ay=t} \left[ \min(1, \mu_P(y)) \right] \\ &= \mu_P\left(\frac{t}{A}\right). \end{aligned} \quad (8.8)$$

## 9. FUZZY DATA ANALYSIS

We will now "fuzzify" the data base  $\{P_{k|h_i}\}_{i=1}^{16}$  introduced in Section 5 for the bootstrap procedure and calculate the vulnerable area  $A_v$  using fuzzy arithmetic. In choosing the membership functions  $\mu_{P_i}$ ,  $i=1, \dots, 16$ , we make use of the fact that the variance of the estimate  $P_{k|h_i}$  is  $P_{k|h_i}(1 - P_{k|h_i})$  although *no statistical properties* will be invoked in the ensuing computation.

Let  $\Delta_i = 1/2 P_{k|h_i}(1 - P_{k|h_i})$ . The membership function for  $P_i$  was taken to be

$$\mu_{P_i}(x) = \begin{cases} 0 & x \leq P_{k|h_i} - 2\Delta_i \\ L_i(x) & P_{k|h_i} - 2\Delta_i < x \leq P_{k|h_i} - \Delta_i \\ 1 & P_{k|h_i} - \Delta_i < x \leq P_{k|h_i} \end{cases} \quad (9.1)$$

where  $L_i(x)$  is the line determined by the points  $(P_{k|h_i} - 2\Delta_i, 0)$  and  $(P_{k|h_i} - \Delta_i, 1)$ ; also, recall that  $\mu_{P_i}(x)$  is symmetric about  $P_{k|h_i}$ .

For example, the fuzzification of  $P_{11} = 0.6$  has as its membership function

$$\mu_{P_{11}}(x) = \begin{cases} 0 & x \leq 0.36 \\ 25x/3 - 3 & 0.36 < x \leq 0.48 \\ 1 & 0.48 < x \leq 0.60 \end{cases} \quad (9.2)$$

From Dubois and Prade or Appendix B the fuzzy number  $P = \sum_{i=1}^{16} P_i$  is given by the membership function

$$\mu_p(x) = \begin{cases} 0 & x \leq a \\ l(x) & a < x \leq b \\ 1 & b < x \leq c \end{cases} \quad (9.3)$$

where

$$\begin{aligned} a &= \sum_{i=1}^{16} (P_{k|h_i} - 2\Delta_i), \\ b &= \sum_{i=1}^{16} (P_{k|h_i} - \Delta_i), \\ c &= \sum_{i=1}^{16} P_{k|h_i}, \end{aligned}$$

and  $l(x)$  is a line segment determined by the points  $(a,0)$  and  $(b,1)$ .

Finally,  $A_v$ , the product of the crisp number  $A$  and the fuzzy number  $P$  has as its membership function

$$\mu_{A_v}(x) = \begin{cases} 0 & x \leq Aa \\ L(x) & Aa < x \leq Ab \\ 1 & Ab < x \leq Ac \end{cases} \quad (9.4)$$

where  $L(x) = l(x/A)$  on  $(Aa, Ab)$ . The membership function for  $A_v$  for the example in Section 3 is given by

$$\mu_{A_v}(x) = \begin{cases} 0 & x \leq 93.7 \\ .051x - 4.78 & 93.7 < x \leq 113.4 \\ 1 & 113.4 < x \leq 133.2 \end{cases} \quad (9.5)$$

as illustrated in Figure 3.

Since the function  $\mu_{A_v}(x)$  is symmetric about  $x = 133.2$ , values of vulnerable area in the interval  $(113.4, 153.0)$ , where the membership function takes on value one, are equally acceptable. The vulnerable area obtained from conventional calculation on the sixteen random shotlines is at the midpoint of this interval, while the true vulnerable area for this example,  $A_v = 105.1$ , has from (9.5) a membership value of 0.58.

Since the fuzzy vulnerable area depends on the cell probabilities, the fuzzy vulnerable area is dependent on the grid size. Grid size impacts the bootstrap and conventional calculations as well. In this example, if the number of cells is increased by a factor of four, the interval of values for which  $\mu_{A_v}(x) = 1$  has as its midpoint the true value.

Furthermore, the interval of values for which the membership function equals one depends upon the interval of values for which the individual cell probabilities have membership values equal to one. We assumed in (9.1) a specific form for the fuzzy cell probabilities. The vulnerability analyst may be able to sharpen these bounds, and in so doing, sharpen the fuzzy vulnerability estimates.

## 10. SUMMARY

In this report, we have attempted to make the point that at many places in vulnerability analysis there is a margin for error that comes about, not from randomness, but from a basic inability to model uncertainty. We have considered, for a specific example, two new approaches for vulnerability analysis: (i) a fuzzy set approach and (ii) a statistical resampling plan known as the bootstrap.

The membership function  $\mu_{A_v}(x)$  in (9.5) for the fuzzy vulnerable area calculation is illustrated in Figure 3. *The fuzzy procedure seeks to quantify the uncertainty inherent in a vulnerable area calculation due to the uncertainty in the cell  $P_{k|h_i}$  estimates.* This uncertainty stems from the assignment of fuzzy kill probabilities to the individual components and from the method of aggregation into a cell probability of kill. It has the distinct advantage, however, of clearly delineating the inherent subjectivity.

The point estimate of  $A_v$  provided by the current method of vulnerability analysis,  $\hat{A}_v = 133.2$ , is indicated in Figure 3. The fuzzy vulnerable area is symmetric about  $\hat{A}_v$ , but considers values in the interval (113.4, 153.0), corresponding to values of  $A_v$  for which the membership function equals one, to be equally acceptable. This interval has the appearance of a statistical confidence interval, but does not convey the same information, since it has no corresponding probability level attached.

As discussed in detail in Section 5, the sampling distribution of  $\hat{A}_v$  provided by the bootstrap has a median of 132.8 and an approximate central 90% confidence interval of [99.3, 163.6]. *The bootstrap procedure seeks to quantify uncertainty inherent in a vulnerable area calculation due to the variability in the sampling procedure responsible for selection of the cell  $P_{k|h_i}$  estimates.* The bootstrap takes the cell  $P_{k|h_i}$  as given, and makes no attempt to model uncertainty in the values themselves.

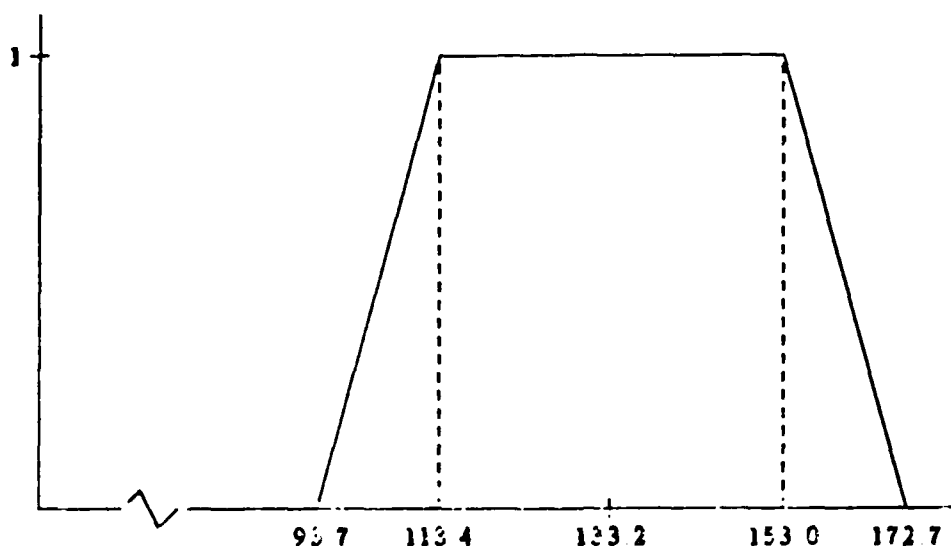


FIGURE 3 MEMBERSHIP FUNCTION FOR FUZZY  $A_v$

The  $A_v$  estimate is subject to error due to the uncertainty in the cell  $P_{k|h_i}$  estimates, the variability in the sampling scheme that selects the cell  $P_{k|h_i}$  estimates, the mesh size of the overlaid target grid, and likely additional sources. The implementation of this observation would suggest bootstrapping on fuzzy numbers. The theoretical development is not sufficient at this time to allow this, but it is a logical research extension. Both the fuzzy set approach and the bootstrap are worthy of implementation into the vulnerability estimation regime. Both procedures enhance the vulnerability estimates as they are now provided without undermining the lore of vulnerability analysis as currently conceived.

In Section 7 the use of fuzzy sets in expressing the uncertainty in the hole area required in a sensitive area of a component to achieve a component kill was illustrated. The fuzzy implication formed by the fuzzy criterion and the fuzzy component kill probability was defined by the cartesian product of the two fuzzy sets (7.1). Other definitions exist for implication and compositional rules; e.g., the recent work of Mizumoto.<sup>13</sup>

<sup>13</sup>M. Mizumoto, "Fuzzy Reasoning Under New Compositional Rules of Inference," *Kybernetes*, 12(1983), 107-117.

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## Appendix A. FUZZY SETS AND NUMBERS

Zadeh,<sup>14</sup> noting that many classes of objects encountered in the real world are not precisely defined, proposed the concept of a fuzzy set to model such classes. The class of all real numbers which are "much greater than one" or the class of "severely damaged vehicles," for example, do not constitute classes in the usual mathematical sense. Fuzzy set methodology holds promise for vulnerability analyses and damage assessments since it provides a mechanism by which subjective information can be quantified and operated on, with results reflecting the degree of imprecision in input information and data. In a closely related field, Blockley<sup>15,16</sup> and Yao<sup>17</sup> have successfully applied fuzzy sets to problems in civil engineering and structural damage.

The fundamental notion of a fuzzy set is the following:

If  $U$  (the universe of discourse) is a collection of objects, then a *fuzzy set*  $A$  is a set of ordered pairs

$$A = \{ (x, \mu_A(x)) \mid x \in U \},$$

where  $\mu_A(x)$ , called a *membership function*, quantifies the grade of membership of  $x$  in  $A$ . The function  $\mu_A(x)$  maps  $x$  into a membership space  $M$  which is usually taken to be the unit interval  $[0,1]$ .

*Example:*  $U$  = Authors of this report;  $A$  = The set of bald men

$$A = \{(\text{Malcolm}, 0.1), (\text{Palmer}, 0.7), (\text{Ralph}, 0.6)\}.$$

For "crisp" sets, i.e., ordinary sets, the membership function reduces to the characteristic function

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases}$$

Fuzzy sets need not be finite sets; for example, in vulnerability studies vehicles take on "loss of function" values in the unit interval  $[0,1]$ . One possibility for a fuzzy set characterizing "high loss of function" is illustrated in Figure 4.

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<sup>14</sup>L. A. Zadeh, "Fuzzy Sets," *Information and Control*, 9(1965), 838-858.

<sup>15</sup>D. I. Blockley, "The Role of Fuzzy Sets in Civil Engineering," *Fuzzy Sets and Systems*, 2(1979), 227-231.

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<sup>17</sup>J. T. P. Yao, "Damage Assessment of Existing Structures," *Journal of the Engineering Mechanics Division, ASCE*, 106(EM4) Paper 15608, (1980), 725-729.

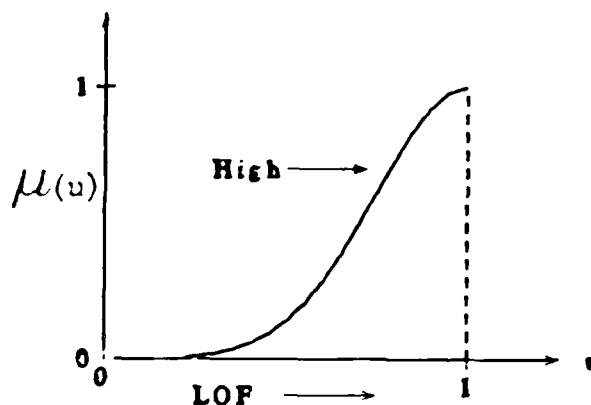


FIGURE 4 MEMBERSHIP FUNCTION FOR "HIGH LOSS-OF-FUNCTION"

We now give several definitions for properties of fuzzy sets which are obvious extensions of corresponding definitions for crisp sets.

A fuzzy set  $A$  is *empty* if and only if

$$\mu_A(x) = 0 \text{ for all } x \in U.$$

Two fuzzy sets  $A$  and  $B$  are *equal* if and only if

$$\mu_A(x) = \mu_B(x) \text{ for all } x \in U.$$

The *complement* of a fuzzy set  $A$ , denoted by  $A^c$ , is defined by

$$\mu_{A^c}(x) = 1 - \mu_A(x) \text{ for all } x \in U.$$

The *union* of two fuzzy sets  $A$  and  $B$ , with membership functions  $\mu_A$  and  $\mu_B$ , is a fuzzy set  $C$  whose membership function is given by

$$\mu_C(x) = \max \{ \mu_A(x), \mu_B(x) \}, \quad x \in U$$

alternatively written

$$\mu_C(x) = \mu_A(x) \cup \mu_B(x), \quad x \in U$$

where  $\cup$  denotes the maximum operator.

The *intersection* of two fuzzy sets A and B is a fuzzy set C given by

$$\mu_C(x) = \mu_A(x) \cap \mu_B(x), \quad x \in U$$

where  $\cap$  denotes the minimum operator.

The *support* of a fuzzy set A is a crisp set S(A), given by

$$S(A) = \{x \mid \mu_A(x) > 0\}.$$

A fuzzy set A is said to be *normal* if and only if

$$\sup_x \mu_A(x) = 1.$$

The *cartesian product*  $A \times B$  of fuzzy sets A and B from universes U and V, respectively, is defined by

$$\mu_{A \times B}(x, y) = \min \{\mu_A(x), \mu_B(y)\}, \quad x \in U, y \in V.$$

A *fuzzy number* A is a fuzzy subset of the real line R.

A fuzzy number A is said to be *convex*, if for any real numbers x, y, z  $\in R$  with  $x \leq y \leq z$

$$\mu_A(y) \geq \mu_A(x) \cap \mu_A(z)$$

An  $\alpha$ -*level set* of a fuzzy number A is a crisp set  $A_\alpha$  defined as

$$A_\alpha = \{x \mid \mu_A(x) \geq \alpha\}, \quad 0 < \alpha \leq 1.$$

The fuzzy set A is commonly written, when U is finite, as

$$A = \mu_A(x_1) / x_1 + \cdots + \mu_A(x_n) / x_n = \sum_i \mu_A(x_i) / x_i$$

where  $\sum$  denotes the union of fuzzy sets; and when U is not finite as

$$A = \int_U \mu_A(x) / x.$$

The algebraic properties of fuzzy numbers under the four arithmetic operations of fuzzy addition, subtraction, multiplication and division have been extensively studied by Mizumoto and Tanaka<sup>18</sup> and Dubois and Prade.<sup>19,20</sup> For example, if A and B are fuzzy numbers with membership functions  $\mu_A$  and  $\mu_B$ , then the sum of A and B is given by

$$\begin{aligned}\mu_{A+B}(z) &= \bigcup_{x+y=z} (\mu_A(x) \cap \mu_B(y)) \\ &= \bigcup_x (\mu_A(x) \cap \mu_B(z-x))\end{aligned}$$

(Recall that  $\bigcup$  and  $\cap$  represent the maximum and minimum operators.)

*Example :* If  $U = \{0, 1, 2, \dots\}$ ,  $A = 0.2/1 + 1/2 + 0.3/3$  and  $B = 0.1/2 + 1/3 + 0.2/4$ , then  $z$  can take the values 3, 4, 5, 6 and 7 and

$$\begin{aligned}\mu_{A+B}(4) &= \max \{ \mu_A(1) \cap \mu_B(3), \mu_A(2) \cap \mu_B(2) \} \\ &= \max \{ 0.2 \cap 1, 1 \cap 0.1 \} = \max \{ 0.2, 0.1 \} \\ &= 0.2.\end{aligned}$$

Fuzzy addition is clearly more cumbersome than crisp addition; however, summation may be readily implemented on the computer.

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<sup>18</sup>M. Mizumoto and K. Tanaka, "Some Properties of Fuzzy Numbers," *Advances in Fuzzy Set Theory and Applications*, M. M. Gupta, R. K. Ragade, R. R. Yager, Editors, North-Holland (1973).

<sup>19</sup>D. Dubois and H. Prade, *op. cit.*

<sup>20</sup>D. Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, (1981).

## Appendix B. A GEOMETRIC INTERPRETATION OF FUZZY ADDITION

In order to determine the sum of  $P_1$  and  $P_2$ , we will use a geometric interpretation to evaluate (8.2). On an  $xy$ -coordinate system, graph  $\mu_{p_1}(x)$ ; for the membership function  $\mu_{p_2}(y)$ , graph the independent variable along the  $y$ -axis as shown in Figure 5. For a given  $t$ , consider the line  $x + y = t$ ; the values that satisfy the equation of this line will be used to evaluate  $\mu_{(p_1 + p_2)}(t)$ .

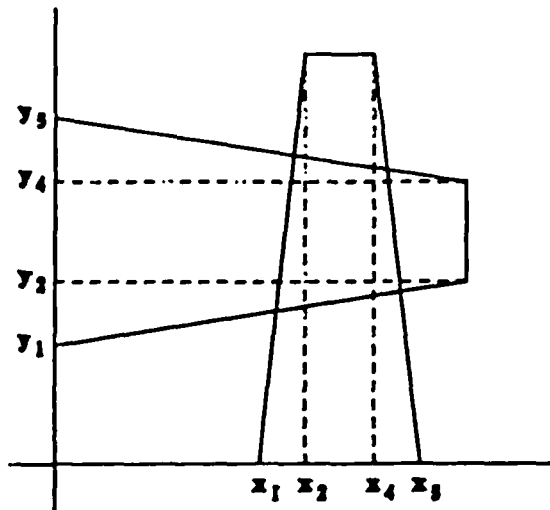


FIGURE 6 MEMBERSHIP FUNCTIONS FOR FUZZY  $P_1$  AND  $P_2$

If  $t \leq x_1 + y_1$  (Figure 6(a)), then either  $\mu_{p_1}(x) = 0$  or  $\mu_{p_2}(y) = 0$ ; this implies

$$\min [\mu_{p_1}(x), \mu_{p_2}(y)] = 0. \quad (\text{B.1})$$

From (8.2),  $\mu_{(p_1 + p_2)}(t) = 0$ . By symmetry, an analogous result holds for  $t \geq x_5 + y_5$ .

Now consider  $t$  in the interval  $x_2 + y_2 < t < x_4 + y_4$ . Then the line  $x + y = t$  passes through the rectangle determined by the line segments  $x_2 x_4$  and  $y_2 y_4$  (Figure 6(b)). At every point  $(x, y)$  in the rectangle, and in particular, at every point on the line  $x + y = t$  within the rectangle, the value of both membership functions is one; therefore,

$$\mu_{(p_1 + p_2)}(t) = 1 \quad \text{for } x_2 + y_2 < t < x_4 + y_4. \quad (\text{B.2})$$

Finally, consider  $t$  in the interval  $x_1 + y_1 < t \leq x_2 + y_2$ . On the line  $x + y = t$  (Figure 6(c)), as  $x$  takes on values zero to  $x_2$ ,  $\mu_{p_1}(x)$  is nondecreasing from zero to one; simultaneously, the values of  $\mu_{p_2}(y)$ ,  $y = t - x$ , are nonincreasing. Thus,

until  $\mu_{p_1}(x) = \mu_{p_2}(y)$ , then

Therefore,

where  $x^*(t)$  satisfies the equation

A similar argument holds for  $x_4 + y_4 \leq t < x_5 + y_5$ .

Suppose  $f(x)$  and  $g(y)$  have the form  $f(x) = a_1x + b_1$  and  $g(y) = a_2y + b_2$ . Then the  $x^*$  that satisfies  $f(x^*) = g(t - x^*)$  is given by

Thus, the sum of two fuzzy numbers from the class (8.4-5) maintains the same analytic structure; that is, line segments are invariant. However, the support is now contained in  $[0, 2]$ . This argument extends naturally to the sum of  $n$  fuzzy numbers.



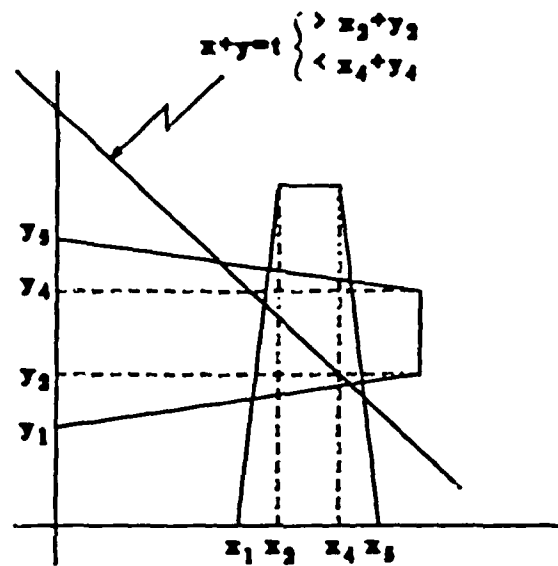


FIGURE 6(b) CASE 2  $x_2 + y_2 < t < x_4 + y_4$

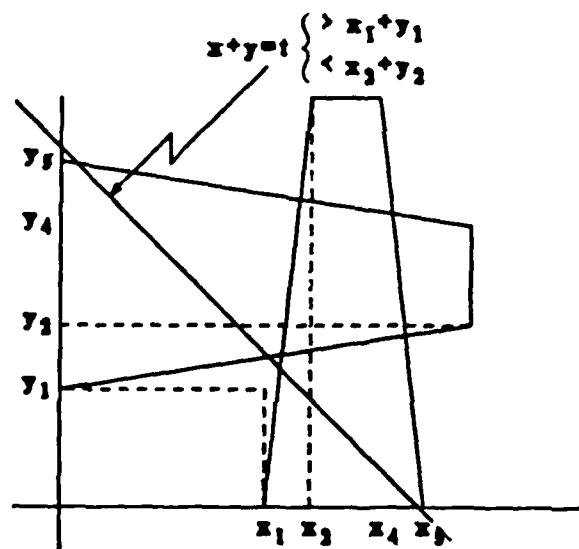


FIGURE 6(c) CASE 3  $x_1 + y_1 < t \leq x_2 + y_2$

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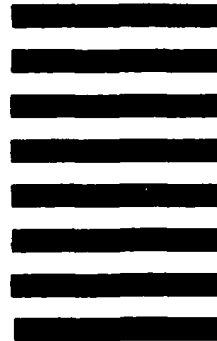


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